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NON-SYMMETRIC KERNELS OF POSITIVE TYPE.

BY DR. CAROLINE E. SEELY.

The properties of continuous symmetric functions of positive type have been studied by Mercer* with especial reference to the uniform convergence of series of their corresponding characteristic functions. In the present paper it is proposed to consider some of the analogous properties of kernels not assumed to be symmetric. Unless otherwise stated, it is assumed that all the kernels $K(s, t)$ considered are real, bounded, and nearly everywhere continuous.

Let us recall some of the well-known theorems for the symmetric case.†

Definition. A function $K(s, t)$ in an interval (a, b) is said to be of positive type if we have

$$(A) \qquad \int_a^b \int_a^b K(s, t)h(s)h(t)dsdt \geq 0$$

for every function $h(s)$ of integrable square.

(I) A necessary and sufficient condition that a symmetric function $K(s, t)$ be of positive type is that its characteristic constants λ_i be all positive.

Corollary 1. If a symmetric kernel is of positive type all its iterated kernels are of positive type.

Corollary 2. All the iterated kernels of even order of any symmetric kernel are of positive type.

(II) The series

$$\sum_{i=0}^{\infty} \frac{\varphi_i(s)\varphi_i(t)}{\lambda_i},$$

where the $\varphi_i(s)$ are the characteristic functions and the λ_i the characteristic constants of any symmetric kernel $K(s, t)$, satisfies the condition of mean convergence.‡

(III) The series

$$\sum_{i=0}^{\infty} \frac{\varphi_i(s)\varphi_i(t)}{\lambda_i},$$

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* Philosophical Transactions, vol. 209 A (1909), p. 415; Goursat, Cours d'Analyse, vol. 3, p. 454.

† Goursat, Cours d'Analyse, vol. 3, p. 449.

‡ Weyl, Mathematische Annalen, vol. 67; Schur, Mathematische Annalen, vol. 66.

where the $\varphi_i(s)$ are the characteristic functions and the λ_i the characteristic constants of a continuous symmetric kernel of positive type, converges uniformly to $K(s, t)$.*

Let the functions $\varphi_i(s)$ and $\psi_i(s)$ be the characteristic functions of a kernel $K(s, t)$ not assumed to be symmetric and let the numbers λ_i be the corresponding characteristic constants. Then we have

$$\int_a^b \varphi_i(s)\psi_i(s)ds = 0 \qquad (\lambda_i \neq \lambda_j).$$

We shall assume the $\varphi_i(s)$ and $\psi_i(s)$ and hence also the λ_i to be all real.

Theorem I. *If $K(s, t)$ is of positive type in an interval (a, b) we have:*

(1) *All the characteristic constants λ_i are positive.*

$$(2) \quad \begin{aligned} \left(\int_a^b \varphi_i(s)\varphi_j(s)ds\right)^2 &\leq \frac{4\lambda_i\lambda_j}{(\lambda_i+\lambda_j)^2}\int_a^b \varphi_i(s)^2ds \cdot \int_a^b \varphi_j(s)^2ds, \\ \left(\int_a^b \psi_i(s)\psi_j(s)ds\right)^2 &\leq \frac{4\lambda_i\lambda_j}{(\lambda_i+\lambda_j)^2}\int_a^b \psi_i(s)^2ds \cdot \int_a^b \psi_j(s)^2ds. \end{aligned}$$

Let

$$h(s) = \alpha\varphi_i(s) + \beta\varphi_j(s).$$

The condition (A) becomes

$$\begin{aligned} &\int_a^b\int_a^b K(s, t)[\alpha\varphi_i(s) + \beta\varphi_j(s)][\alpha\varphi_i(t) + \beta\varphi_j(t)]dsdt \\ &\quad = \int_a^b \left[\frac{\alpha\varphi_i(s)}{\lambda_i} + \frac{\beta\varphi_j(s)}{\lambda_j}\right][\alpha\varphi_i(s) + \beta\varphi_j(s)]ds \\ &\quad = \frac{\alpha^2}{\lambda_i}\int_a^b \varphi_i(s)^2ds + \alpha\beta\left(\frac{1}{\lambda_i}+\frac{1}{\lambda_j}\right)\int_a^b \varphi_i(s)\varphi_j(s)ds + \frac{\beta^2}{\lambda_j}\int_a^b \varphi_j(s)^2ds \geq 0, \end{aligned}$$

and this condition must hold for every choice of α and β. That is, the quadratic form in α and β must be semi-definite, and the conditions for semi-definiteness are

$$\frac{1}{\lambda_i}\int_a^b \varphi_i(s)^2ds > 0, \qquad \frac{1}{\lambda_j}\int_a^b \varphi_j(s)^2ds > 0.$$

$$\left[\left(\frac{1}{\lambda_i}+\frac{1}{\lambda_j}\right)\int_a^b \varphi_i(s)\varphi_j(s)ds\right]^2 \leq 4\frac{1}{\lambda_i}\int_a^b \varphi_i(s)^2ds \cdot \frac{1}{\lambda_j}\int_a^b \varphi_j(s)^2ds.$$

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* See Mercer, loc. cit. The assumption of continuity here is essential. In fact, Toeplitz has constructed a bounded symmetric kernel of positive type, continuous except at one point, and such that the series

$$\sum_{i=1}^{\infty} \frac{\varphi_i(s)\varphi_i(t)}{\lambda_i}$$

is not uniformly convergent. See Mathematische Abhandlungen Hermann Amandus Schwarz gewidmet, p. 426.

From these follow

$$\lambda_i > 0, \qquad \lambda_j > 0,$$

$$\left[\int_a^b \varphi_i(s)\varphi_j(s)ds\right]^2 \leq \frac{4\lambda_i\lambda_j}{(\lambda_i+\lambda_j)^2}\int_a^b \varphi_i(s)^2ds\int_a^b \varphi_j(s)^2ds.$$

The second condition of (2) is obtained in exactly similar manner by letting $h(s) = \alpha\psi_i(s) + \beta\psi_j(s)$.

It has just been shown that a necessary condition for a kernel $K(s, t)$ to be of positive type is that the characteristic constants be all positive. It is not, however, true that this condition is also sufficient, as is the case with symmetric kernels. This can be seen from the following example. Let

$$K(s, t) = \frac{\sin s\,(\sin t + \sin 3t)}{\sqrt{\pi}} + \frac{\sin 2s \sin 2t}{2\sqrt{\pi}}.$$

This kernel has evidently, for the interval $(-\pi, \pi)$, the characteristic constants $\lambda_1 = 1$, $\lambda_2 = 2$ corresponding to the characteristic functions

$$\varphi_1(s) = \frac{\sin s}{\sqrt{\pi}}, \qquad \varphi_2(s) = \frac{\sin 2s}{\sqrt{\pi}},$$

$$\psi_1(s) = \frac{\sin s + \sin 3s}{\sqrt{\pi}}, \qquad \psi_2(s) = \frac{\sin 2s}{\sqrt{\pi}},$$

and no others. Let $h(s) = \sin s + \alpha \sin 3s$, where α is a constant. Then

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K(s, t)h(s)h(t)dsdt = \pi^{3/2}(1+\alpha),$$

which may be made positive or negative by suitable choice of α.

Theorem II. *If a kernel $K(s, t)$, with more than one characteristic constant, is such that for every positive number n there exists an iterated kernel $K^m(s, t)$, $m > n$, that is of positive type, then if the functions $\varphi_i(s)$, $\psi_i(s)$ are characteristic functions of $K(s, t)$ we have*

$$\int_a^b \varphi_i(s)\varphi_j(s)ds = 0,$$

$$\int_a^b \psi_i(s)\psi_j(s)ds = 0 \qquad (\lambda_i \neq \lambda_j),$$

that is, the $\varphi_i(s)$ and $\psi_i(s)$ each form an orthogonal set.

For the functions $\varphi_i(s)$ and $\psi_i(s)$ are also characteristic functions of the iterated kernel $K^m(s, t)$, corresponding to the characteristic constants λ_i^m. Then from Theorem I (2) we have

$$\left[\int_a^b \varphi_i(s)\varphi_j(s)ds\right]^2 \leq \frac{4\lambda_i^m\lambda_j^m}{(\lambda_i^m+\lambda_j^m)^2}\int_a^b \varphi_i(s)^2ds\int_a^b \varphi_j(s)^2ds,$$

and it is clear that for $\lambda_j > \lambda_i$ the first factor of the right member approaches zero as m increases indefinitely. Whence the constant of the left member is zero, and the functions $\varphi_i(s)$ form an orthogonal set. The same argument holds for the $\psi_i(s)$.

It may be noted that the hypothesis of Theorem II is satisfied by all symmetric kernels, since all their iterated kernels of even order are of positive type (see Corollary 2 of the known theorem (I)).

It might be expected from this theorem that a kernel whose characteristic constants are all positive and whose characteristic functions $\varphi_i(s)$ and $\psi_i(s)$ form two orthogonal sets must be of positive type. That this is not the case is shown by the example given on page 174, for there we have

$$\int_a^b \varphi_1(s)\varphi_2(s)ds = 0, \qquad \int_a^b \psi_1(s)\psi_2(s)ds = 0, \qquad \lambda_1 = 1, \qquad \lambda_2 = 2$$

and yet the kernel is not of positive type.*

Theorem III. *Let* $K(s, t)$ *be a kernel satisfying the hypothesis of Theorem II, and let the functions* $\varphi_{i1}(s), \cdots, \varphi_{im_i}(s)$ *and* $\psi_{i1}(s), \cdots, \psi_{im_i}(s)$ *(where* m_i *is the number of independent solutions of the homogeneous integral equation corresponding to the characteristic constant* λ_i*) be a set of its characteristic functions so chosen that*

$$\int_a^b \varphi_{ij}(s)\varphi_{ik}(s)ds = 0, \qquad \int_a^b \psi_{ij}(s)\psi_{ik}(s)ds = 0 \quad (j \neq k).$$

Suppose moreover

$$\int_a^b \varphi_{ik}(s)^2ds < N, \qquad \int_a^b \psi_{ik}(s)^2ds < N,$$

where N *is some number independent of* i *and* k*. Then the series*

$$\sum_{i=1}^{\infty}\frac{\varphi_{i1}(s)\psi_{i1}(t)+\varphi_{i2}(s)\psi_{i2}(t)+\cdots+\varphi_{im_i}(s)\psi_{im_i}(t)}{\lambda_i}$$

satisfies the condition of mean convergence.

This theorem is an immediate consequence of Theorem II, of the definition of mean convergence, and of Schur's theorem.† For

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* I am indebted to Dr. Gronwall for this example, as well as for other welcome suggestions.

† The series $\sum_{i=1}^{\infty}\frac{1}{\lambda_i^2}$, where the λ_i are characteristic constants of some kernel $K(s, t)$ each repeated m_i times, is absolutely convergent since $m_i \leq$ the order of λ_i. See Schur, loc. cit., p. 508.

$$\lim_{\mu,\,\nu=\infty}\int_a^b\int_a^b\left[\sum_{i=1}^{\mu+\nu}\frac{\varphi_{i1}(s)\psi_{i1}(t)+\cdots+\varphi_{im_i}(s)\psi_{im_i}(t)}{\lambda_i}\right.$$

$$\left.-\sum_{i=1}^{\mu}\frac{\varphi_{i1}(s)\psi_{i1}(t)+\cdots+\varphi_{im_i}(s)\psi_{im_i}(t)}{\lambda_i}\right]^2 dsdt$$

$$=\lim_{\mu,\,\nu=\infty}\sum_{i=\mu}^{\mu+\nu}\int_a^b\int_a^b\frac{\varphi_{i1}(s)^2\psi_{i1}(t)^2+\cdots+\varphi_{im_i}(s)^2\psi_{im_i}(t)^2}{\lambda_i^2}dsdt$$

$$\leq\lim_{\mu,\,\nu=\infty}\sum_{i=\mu}^{\mu+\nu}\frac{N^2m_i}{\lambda_i^2}=0,$$

that is, the series is convergent in the mean. The final step is from Schur's theorem.

It will be noted that in the proof of Theorem I we did not make full use of the positive character of $K(s, t)$, but only of the fact that condition (A) was satisfied for the restricted class of all functions that are of the form $\alpha\varphi_i(s)+\beta\varphi_j(s)$. It is evident that Theorems II and III are also true under this much weaker assumption.